**Renormalization Group**

And now we’ll look at renormalizing the field theory for the Ising ferromagnet. This will provide a better understanding of the whole technique.

**RG analysis of Ising model in arbitrary d**

So we consider again the Ising model.



where,



So we want to go from a system defined on a lattice spacing *a*, to one defined on a lattice spacing b*a*. The original lattice spacing would hold kurvatures between (0, Λ = 2π/a) [though, in our units, a = 1], and the new one kurvatures between (0, Λ/b). So to rescale we want to take our system and split the kurvatures into (0, Λ/b) + (Λ/b, Λ). The former we’ll call long wavelength modes, and the latter short wavelength modes. We will want to integrate out the short ones: (Λ/b, Λ). So we make the separation:



where,



So going to Fourier space (see Fourier transform file),



Then we split φ(k) into φshort(k) + φlong(k). Note that certain cross terms will vanish. For instance ∫dkφshort(k)φlong(k) must be zero, because the k in dk cannot be both short and long. However certain cross terms can exist in the interaction term since k1 could be short, and k2 long for instance. So we’ll get something like:



where *perm* means we are permuting the order of the indices 1,2,3,4. Since all the permutations are identical, when integrated, we can say,



One more thing. We’ll recognize the first term in the bracket as the bare u-vertex for the short φ’s, and the last term in the bracket as the bare u-interaction vertex for the long φ’s. Grouping things together, we have:



I’m going to call the first line S[φℓ], and the second line S[φs], and the third line Sint[φs,φℓ]. Now we want to integrate out the short guys. This can’t be done exactly, but can be done perturbatively in powers of u. Stepping back a bit, this will entail the following manipulation. Might check out the Interacting electrons + phonons / Superconductors / Thermal Properties / GF Free Energy for another example of this sort of thing.



The outside blue integral is just a constant and so can be discarded. So we have:



where of course,



Then once we calculate < >short, will take its ln and put it in the exponent, and then try to work out the new renormalized action in terms of φℓ d.o.f.

**Feynman Rules**

Okay so now we’ll write down the rules for perturbatively calculating those diagrams that contribute to the < >short thing above. And note u/4 = 6u/4!. So,

A picture containing text, watch

Description automatically generated

The dotted lines represent φs, and the solid lines φℓ. The solid φℓ lines are basically external lines as far as φs is concerned. Momentum is conserved at each vertex. The usual symmetry factors apply, as discussed in the path integrals folder for instance. Then we sum (1/Ld)Σk or integrate ∫ddk/(2π)d over all momenta, including those on the φℓ’s, but noting that the k’s in a φs will range from Λ/b → Λ, while k’s in a φℓ will ange from 0 → Λ/b. Well might note that since we factored the lattice spacing, a*,* out of our x coordinate in the action, the sum would really be (1/N)Σk since k is now unitless. Out to O(u2), and neglecting j-terms, the diagrammatic expansion would look like (I’ve read that we don’t need to consider disconnected diagrams – these will cancel out, ultimately, when exponentiate into S),

A picture containing clock, several

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The last diagram is actually zero because by momentum conservation, the momentum flowing in from the left on the long leg, must be the same as the momentum flowing through the middle short leg, but long legs and short legs can’t have the same momentum. And actually, there is one more O(u2) diagram – one with 6 φℓ external legs. This would contribute to a renormalization of a potential φ6 term in the action we started with, if we had such a term. But we’ll see that these guys scale away to a constant; so we don’t need to include them as they’re ‘marginally’ relevant.

**O(u0) calculation**

So let’s take the first term in the expansion,

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Then we have:



So then now we need to see how we must rescale our φℓ variables to put Z back into its original form vis a vis the momentum integration. So apropos S(φℓ), we have:



First we need to rescale the momentum:



and we’ll get:



Then we can redefine the dummy variable of integration inside D[φℓ(k)] to be φℓ(k/b) → φℓ(k). There would be a Jacobian associated with this change of variables, but it would just depend on b, and be a multiplicative constant for Z, and an additive constant for F, and so wouldn’t matter in the end. Also going to factor out that b from the δ function, which gives us an additional factor of bd, since δ(f(x)-f(a)) = δ(x-a)/|f´(a)|.



Now we need this form to match the old one, but with renormalized coupling constants. So we need to rescale the φ’s to take the net 1/b(2+d) factor off of the k2 term in S0. So we absorb the b-terms into the φℓ term via:



This will introduce another Jacobian factor in D[φ] of roughly b(1+d/2)Λ but this doesn’t matter, again. So we have:



Let’s specialize to constant fields. Then j(k/b) will be ∫d3x je-i**k**/b·**x** = (2π)djδ(k/b) = (2π)djbdδ(k) = bdj(k), where j(k) = (2π)djδ(k). So then we have:



Converting back to position space, we have:



So now it’s back in the desired form, and with renormalized coupling constants. So our renormalized terms so far would be:



So at this order, we have:



So in d > 4, u(bL) would scale away to irrelevance, and we’d have only two relevant couplings – the temperature one, and the magnetic field one, as Widom says we should. So what would be our critical exponents at this point? Well, accounting for the fact that we have the dangerously irrelevant variable u, and using the equations in the RG file (first one) we have:



We can use the critical exponent relations to get the others. So everything is checking out as we are recovering the mean field exponents. So basically, in d > 4 mean field theory works, and this is because the fluctuations induced by the u term are rendered insignificant thanks to the averaging power of the higher spatial dimensions (or something?) Looks like we’ll then find all of the Sint terms scale away to irrelevance when d > 4. Okay, so on to it…

**O(u1,2) calculation**

Let’s go out to out to one order in u – I mean we’ll include diagrams which will renormalize the vertices by one order in u (think this is called one-loop order). Again, I’m going to ignore j-diagrams. So we’ll look at:

A picture containing clock, watch

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The first diagram is (note symmetry factor of ½ for coincident propagator):



And let’s see what will happen to the Sint(φs,φℓ) term when we rescale as we did in the previous section. So let’s do for the φℓ’s: k → k/b, φℓ(k/b) → φℓ(k), φℓ(k) → b1+d/2φℓ(k).



Converting back to position space, and eliminating the ℓ subscript, ‘cause again our range of momentum integration is now entire.



Okay what about the middle term? We have:



The symmetry factor comes from two coincident propagators (so (1/2)2) and two identical propagators (so 1/2!). Well this is just a constant; there are no φℓ’s to rescale. So just have:



And now let’s examine the last term,



where the 2!2! comes from the identical propagators and identical points. Now let’s rescale the φℓ guys: k → k/b, φℓ(k/b) → φℓ(k), φℓ(k) → b1+d/2φℓ(k). So,



In the large b limit, this comes to:



So now that that integral is decoupled from the φℓ integral, we can put the φℓ integrals back into position space, and we’ll drop the ℓ subscript since the integration is entire.



Combining everything, we now have:



And let’s fill this into our Z,



So the effective action so far is, dropping that irrelevant constant term in Z1:



So to O(u) our RG equations are:



Might notice that the b2 and b4-d prefactors all come from the kℓ and φℓ rescaling, not from ks or φs. Might observe at this point that in d > 4, the u(bL) renormalizes to zero and becomes irrelevant. Well it is easier to solve these equations if we convert them to differential equations. We can do this because they are true for all b and L. So let b = 1+ε, and expand the equations out to first order in ε. I’ll do this with the first equation too, just for comparison’s sake. So,



The solution is:



And we can see this satisfies the original equation, as:



Now let’s look at the r-equation [Ωd is surface area of d-dimensional sphere – see Appendix]



and the same sort of thing can be done for u,



Defining Sd = Ωd/(2π)d, we can write these as:



So there we are. Well, I’m going to write these with b as the independent variable, rather than L. Basically just renaming the independent variable. And we’ll implicitly keep L fixed to the original lattice spacing size.



**Analysis d > 4**

Now we like to look for fixed points. j\* = 0 is one. As for the other two, we need to solve:



Obvious solutions are r\* = u\* = 0.



And linearizing the equations about this point would give us:



and,



So our linearized equations are:



We can solve these coupled equations. Could solve the bottom and then plug into the top. But I’ll use the general method outlined in the RG file. So we’ll write:



As worked out in the RG file appendix, the solution is:



where λ and |λ> are the eigenvalues/eigenvectors of , which are:



So in our case, we have:



And so the solution is:



And separately,



Of course Δr = r and Δu = u since fixed point is at r\* = u\* = 0. For d > 4, the large b limit will make u-dependent terms scale to obselescence. And the r will scale as b2. This is the same as our mean field theory results. We can read off the exponents,



This gives us ν = 1/λr = ½, and Δ = λj/λr = ½ + d/4, and all the MF exponents follow. So we see again that when d > dupper\_critical, MFT works just fine. We can plot the fixed point, eigenvectors, and typical flows. Looks like this. Flows in the j direction can’t really be shown, but just know that the arrows would flow out/into the page as well, according to whether they start above, below the j = 0 plane.

Chart

Description automatically generated with medium confidence

The vector along r-axis is |λ1>. The vector along the dotted line is |λ2> (its first component is along r and second along u – so components read a little backwards, and the arrow isn’t the direction of the vector – just the direction of the flows in its vicinity). And the third eigenvector (dot sticking out of the page) is along the j-axis. From our discussion in the RG file, we know that |λ2> points along the critical hypersurface, at least near the fixed point, since it’s the one which corresponds to a negative eigenvalue. We can find the corrections to the critical temperature for a given u. It’s just the projection of the u value down to the critical curve. Since |λ2> is given by:



(first component is along r-axis, and second component along u-axis) rc near the fixed point would be given by:



And we have:



So we can see fluctuations decrease the critical temperature, as expected. Might compare this to what we found from our self-consistent HF approximation to the GF in the Ising MF+Fluctions file,



and we see this actually matches (!). I guess that isn’t too surprising, because for d > duc = 4, the u term scales to irrelevance, i.e., fluctuations are small, and so our perturbative calculations should match up with RG. But still surprising that our RG matches up to the SCHF approach, which implicitly summed up an infinite number of diagrams.

**Analysis d < 4**

Now if d < 4 we see that ub will not scale to irrelevance, because its eigenvalue 4-d > 0. But only r and j should be relevant variables. So there must be another fixed point near which ub’s eigenvalue becomes negative again. Let’s look for it. We do so perturbatively by treating 4 – d = ε as a small parameter. I suppose this is because we know that the critical point and exponents must reduce to the MF values when ε = 0. And we can presume these values to depend analytically on ε, i.e., a Taylor series would be appropriate. In contrast, u’s importance grows as we get close to the critical point, and that is why the perturbation theory in u approach that we took in files past, doesn’t work so well.



So our fixed point is:



Since Λ = 2π, and we’ll say z = 6 for a cubic lattice, r\* ~ -ε itself. Apropos u\*, 1/SdΛd-4 = Λ4/Ωd = in 3D … (2π)4/(4π) = 4π3. And so u\* ~ 4π3ε/9z2 ~ 4πε/z2 ~ ε/3. Now let’s linearize the equations about this point. So we’ll introduce variables Δu = u – u\*, and Δr = r – r\*.



And let’s keep just O(ε) terms,



And now the other one,



And expanding to first order in ε,



So altogether, we have:



Filling in r\* and u\*, oh and replacing r and u with Δr, and Δu, in LHS of equation,



Now let’s solve these. Again I’ll avail myself of the general results in the RG file appendix to solve this system of equations. So we’ll write:



As worked out in the RG file appendix, the solution is:



where λ and |λ> are the eigenvalues/eigenvectors of , which are, generically:



And in our present case these work out to:



And so the solution is:



And separately,



And now for d < 4, i.e., ε > 0, the u term will scale to irrelevance as required, though not dangerous irrelevance, since u\* is non-zero. And the r will scale as b2-ε/3. So our λr is now 2-ε/3. So, along with λj = 1+d/2 = 1+(4-ε)/2 = 3 – ε/2, we have:



solving for ν and Δ, we get:



and so from these two exponents, λr and λj we can obtain the rest of the critical exponents. We find, using various relations from the RG file.



ε = 1 would correspond to 3D. And let’s compare to the actual values in 3D,

Table

Description automatically generated

So much better agreement! We can plot the fixed point, eigenvectors, and typical flows. Looks like this. Flows in the j direction can’t really be shown, but just know that the arrows would flow out/into the page as well, according to whether they start above, below the j = 0 plane.

Diagram

Description automatically generated

The fixed point is the red dot at the intersection of the dotted line and vertical line. The eigenvectors are drawn emanating from the fixed point. The vector pointing along r-axis is |λ1>. The vector along the dotted line is |λ2>. And the third eigenvector (dot sticking out of the page right on top of the fixed point) is along the j-axis. From our discussion in the RG file, we know that |λ2> points along the critical hypersurface, at least near the fixed point, since it’s the one which corresponds to a negative eigenvalue. We can see that above the critical hypersurface, the metal scales towards the high T paramagnetic phase, and below the surface, scales towards the low T ferromagnetic phase. We can find the corrections to the critical temperature for a given u. It’s just the projection of the u value down to the critical curve. Since |λ2> is given by:



(first component is along r-axis, and second component along u-axis) rc near the fixed point would be given by:



And we have:



This doesn’t seem to differ too much from the d > 4 value, for ε = 1, but it *is* different. Well, it seems I presumed the critical hypersurface could be approximated as a straight line to through the origin.

**Comment about relevant perturbations and going from discrete to continuum limit**

We saw that in d > 4, the φ4 term was irrelevant. This can be ascertained earlier in our analysis, without going through all the <Sint> perturbative stuff. Once we’ve worked out the requisite scaling requirements for k, and φ, in S0, we can test any term in the original unscaled H for relevance by making those scaling manipulations to the perturbative term in question. And if the terms scale to 0, we can neglect them. For instance, we would just do:



and see that it doesn’t matter in d > 4. I think we will find likewise that terms φn terms that could come about due to rescaling will also go to zero. These will, for instance, look something like this:



and under rescaling k → k/b, φℓ → b1+d/2φℓ, we have:



So terms only matter if:



For the different dimensions,



So trying to use this model for d = 2 is probably not a good idea, as basically all φn terms would be relevant. For d = 3, the terms allowed would be φ4, φ5 and φ6. We include the φ4 of course. But the φ5 term could be disallowed on symmetry grounds, while the φ6 term would be marginally relevant. It’s scaling exponent would be n + d – dn/2 = 6 + 3 – 3·6/2 = 0. So it would neither grow nor shrink under rescaling. I think this means that it *can* have a subdominant effect on the scaling. For d = 4, the max allowed term would be φ4.

Another thing: remember in files back that when we were translating the Ising model to the continuum limit, we left out certain terms, like kn>2 terms. We could ask whether these are important close to the critical point. In general, the rescaling should tell us. For instance,

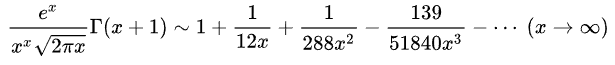


and is thus irrelevant in all dimensions.

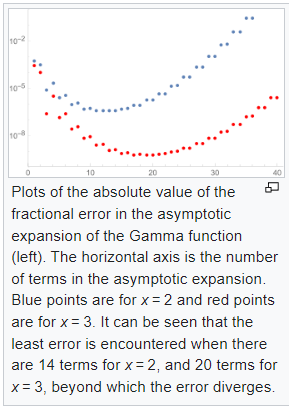
And we’ll observe that the field rescaling factor we’re using, ζ = b1+d/2, was determined already at the level of the free theory. At least in the large b limit, this feature persists for the NLσM. Maybe it always will? So it seems that determining ζ at the level of the free theory suffices insofar as seeing which terms in S will survive renormalization in the large b limit. And note ζ doesn’t depend on the fixed point or anything like that.

**ε series is asymptotic**

Evidently the series generated here is asymptotic. And so for a fixed ε, like say 1, additional terms in the series will improve convergence up to a point, but after which start to diverge. An example is the Stirling approximation to the gamma function:



Apparently, for a fixed x, keeping more and more terms will improve agreement, but eventually it will begin to diverge once we get past the nmax term. If we know the full asymptotic expansion formula, then we can see what nmax is. But generally speaking, nmax will depend on x, and the larger x is, the larger nmax will be too. So one could say that as x → ∞, nmax → ∞. Stolen from Wikipedia. Can see that nmax increases with x.



Still, one can apply Borrel summation techniques to resum the series and improve convergence.

**Example**

Say we have RG equations,



What is the critical temperature at u = -1? We have:



As worked out in the RG file appendix, the solution is:



where λ and |λ> are the eigenvalues/eigenvectors of , which are:



Eigenvectors are given by:



So,



Chart

Description automatically generated with medium confidence

We can find the corrections to the critical temperature for a given u. It’s just the projection of the u value down to the critical curve, which is the line along the negative valued eigenvector. Since |λ2> is given by:



(first component is along r-axis, and second component along u-axis) rc near the fixed point would be given by:



and using u = -1, we have:

.

**Appendix**

We can get the volume of a d-dimensional unit sphere. Saw this in C. Van Vliet’s book. So consider a d-dimensional Gaussian integral, and we’ll change to d spherical coordinates in the second line,



So we have:



Just to check, in 3D, we should have Ω3 = 4π:



So that checks out.